Math 113 (Calculus II)
Exam 2 RED KEY

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided. Responses written on your exam will be ignored.*

1. Evaluate $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{2}}} \frac{1}{x^2\sqrt{x^2-1/4}}\,dx.$

   a) $1-\sqrt{3}$ b) $2-2\sqrt{3}$
   c) $-1+\sqrt{3}$ d) $-2+2\sqrt{2}$
   e) $1-\sqrt{2}$ f) $1-2\sqrt{2}$

   Solution: The answer is (d)

2. What is the value of $\displaystyle\int_0^1 \frac{1}{x^2-5x+6}\,dx$?

   a) $\ln 3$ b) $\ln 2-\ln 3$
   c) $\ln 3-2\ln 2$ d) $\ln 2$
   e) $2\ln 2-\ln 3$ f) none of the above

   Solution: The answer is (e)

3. Find $\displaystyle\int_0^{\sqrt{2}} x\sqrt{4-x^4}\,dx.$

   a) $\sin^{-1}(\sqrt{2})$
   b) $\pi/8$
   c) $\pi/4$
   d) $\pi/2$
   e) π
   f) 2π

   Solution: The answer is (c)

4. Suppose $f(x)$, $0\le x\le 1$, is four times differentiable and satisfies $|f^{(4)}(x)|\le 90$ for $0\le x\le 1$. Then the error E_S of Simpson's Rule S_{2n} for the approximation of $\int_0^1 f(x)\,dx$ satisfies

   a) $|E_S|\le 1/2n^4$,
   b) $|E_S|\le 1/2n^2$,
   c) $|E_S|\le 15/2n^4$,
   d) $|E_S|\le 15/2n^2$,
   e) $|E_S|\le 15/4n^4$,
   f) $|E_S|\le 15/4n^2$.

   Solution: The answer is (a)

5. Which of the following three improper integrals are convergent?

$$\text{I} \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx \quad \text{II} \int_0^\infty \frac{\arctan(x)}{2+e^x}\,dx \quad \text{III} \int_2^\infty \frac{2+e^{-x}}{x}\,dx$$

a) none of them	b) I only	c) II only
d) III only	e) I and II only	f) I and III only
g) II and III only	h) all of them	

Solution: The answer is (c)

6. Find the arc length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

   a) 0
   b) $\ln(2+\sqrt{3})$
   c) $\ln(\pi/3)$
   d) $\sqrt{\pi/3}$
   e) $1/2$
   f) $e^{\pi/3}$

Solution: The answer is (b)

7. Find the centroid of the region between the curve $y = \sin(5x)$ and the x-axis for $0 \le x \le \pi/5$.

   a) $\left(\frac{\pi}{10}, \frac{\pi}{8}\right)$
   b) $\left(\frac{\pi}{10}, \pi\right)$
   c) $\left(\pi, \frac{\pi}{8}\right)$
   d) $\left(-\frac{\pi}{10}, \frac{\pi}{8}\right)$
   e) $\left(-\frac{1}{10}, \frac{1}{8}\right)$
   f) none of the above

Solution: The answer is (a)

Part II: *In the following problems, show all work, and simplify your results.*

8. (8 points) Evaluate $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$.

Solution: Let $x = \sec\theta$, with $dx = \sec\theta\tan\theta\,d\theta$. Then,

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int \tan^2\theta\,d\theta$$

$$= \int sec^2\theta - 1\,d\theta = \tan\theta - \theta + C.$$

Since $x = \sec\theta$, it is easy to see that $\tan\theta = \sqrt{x^2-1}$. Thus, the integral is

$$\sqrt{x^2-1} + \sec^{-1}(x) + C.$$

9. (10 points) Find $\displaystyle\int \frac{x^4+1}{x^3+x}\,dx$.

Solution: First, divide:

$$\begin{array}{r|l} & x \\ x^3+x) & x^4 \qquad +1 \\ & -x^4 - x^2 \\ \hline & -x^2 \end{array}$$

Hence,

$$\frac{x^4+1}{x^3+x} = x + \frac{1-x^2}{x^3+x}.$$

By partial fractions,

$$\frac{1-x^2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

or

$$1 - x^2 = A(x^2+1) + (Bx+C)x.$$

If $x = 0$, we see that $A = 1$. If $x = 1$, then $2A + B + C = 0$, or $B + C = -2$. If $x = -1$, we see that $2A + B - C = 0$, or $B - C = -2$. Thus, $B = -2$ and $C = 0$. Thus,

$$\int \frac{x^4+1}{x^3+x}\,dx = \int \left(x + \frac{1}{x} - \frac{2x}{x^2+1}\right)\,dx = \frac{x^2}{2} + \ln|x| - \ln(x^2+1) + C.$$

10. (10 points) Find $\int \frac{e^x+1}{e^x-1}\,dx$. [Hint: try a substitution.]

Solution: Let $u = e^x$. Then, $du = e^x\,dx$, or $du = u\,dx$. Thus, $du/u = dx$.

$$\int \frac{e^x+1}{e^x-1}\,dx = \int \frac{u+1}{u-1}\frac{du}{u}.$$

By partial fractions,

$$\frac{u+1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1},$$

or

$$u + 1 = A(u-1) + Bu.$$

If $u = 0$ we see that $A = -1$. If $u = 1$, then $B = 2$. So,

$$\int \frac{u+1}{u-1}\frac{du}{u} = \int \left(2\frac{1}{u-1} - \frac{1}{u}\right)\,du = 2\ln|u-1| - \ln|u| + C$$

$$= 2\ln|e^x - 1| - x + C.$$

11. (10 points) Use the trapezoidal rule T_n with $n = 3$ to estimate the value of

$$\int_0^3 (x^3 - x^2 + x - 1)\,dx.$$

Solution: If $f(x) = x^3 - x^2 + x - 1$, then since $\Delta x = 1$, The trapezoid rule becomes

$$\frac{1}{2}(f(0) + 2f(1) + 2f(2) + f(3)) = \frac{1}{2}(-1 + 2\cdot 0 + 2\cdot 5 + 20) = \frac{29}{2}.$$

12. (7 points) Show that $e^{x+y} = e^x e^y$ for all real numbers x,y. Hint: Use the fact that ln is a one-to-one function.

Solution:

$$\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x + y = \ln(e^{x+y}).$$

Since ln is a 1-1 function,

$$e^x e^y = e^{x+y}.$$

13. (10 points) Find the area of the surface obtained by rotating the curve $x = 1+2y^2$, $1 \le y \le 2$, about the x-axis.

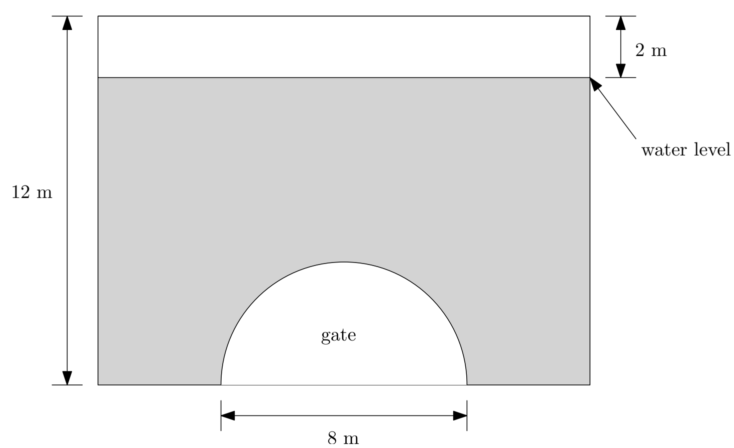
Solution: Note $\frac{dx}{dy} = 4y$, so

$$V = \int_1^2 2\pi y\sqrt{1+16y^2}\,dy.$$

By letting $u = 1+16y^2$, $du = 32y\,dy$, and the above integral becomes

$$\frac{\pi}{16}\int_{17}^{65} u^{1/2}\,du = \frac{\pi}{16}\cdot\frac{2}{3}u^{3/2}|_{17}^{65} = \frac{\pi}{24}(65^{3/2} - 17^{3/2}).$$

14. (10 points) A 12 meter high vertical dam in the shape of a rectangle has an underwater semicircular gate as shown in the figure. Find the hydrostatic force against the gate. (Assume that the density of water is 1000 kg/m^3 and that the acceleration due to gravity is 9.8 m/s^2.)



Solution: If we set $y = 0$ to be at the top of the water, then the width across the plate at depth y is $2\sqrt{16-(10-y)^2}$. Thus, the force due to fluid pressure is

$$F = \int_6^{10} 9800y\cdot 2\sqrt{16-(10-y)^2}\,dy.$$

If we use a change of variables $u = 10-y$, then $du = -dy$, and the above integral becomes

$$F = \int_4^0 9800(10-u)2\sqrt{16-u^2}\,(-du) = 2\cdot 9800\int_0^4 (10-u)\sqrt{16-u^2}\,du$$

(which incidentally, is the integral when you set $y = 0$ at the bottom of the gate). Notice that the above integral becomes

$$196000\int_0^4 \sqrt{16-u^2}\,du + 9800\int_0^4 2u\sqrt{16-u^2}\,du.$$

The first integral is recognizable as a quarter circle of radius 4, while the second can be done with substitution. the above integral then becomes

$$196000\cdot 4\pi - 9800\cdot\frac{2}{3}(16-u^2)^{3/2}|_0^4 = 196000\cdot 4\pi - \frac{19600}{3}(64)$$